GRAVITATION, GRAVITY AND PLANATORY MOTION

Kepler's Law of Planatory Motion

(i) **Law of Orbit :** Every planet moves in an elliptical path or orbit, with the sun at one of the focii of the ellipse.

(ii) **Law of Areas :** The line joining the sun and the planet sweeps out equal areal in equal interval of time.

or

The areal velocity of a planet around the sun is a constant.

(iii) **Law of periods :** The square of the time-period of a planet around the sun is proportional to the cube of the semi-major axis, or

$$T^2 \propto a^3$$

Newton's Universal Law of Gravitation : Newton stated the law of universal gravitation.

The gravitational force between any two objects of masses m_1 and m_2, separated by a distance r is attractive in nature, acting along the line joining the two objects and its magnitude is

(i) directly proportional to the product of their masses, i.e., $F \propto m_1 m_2$. and

(ii) inversely proportional to the square of the distance between them, i.e., $F \propto \frac{1}{r^2}$

Combining these two statements together, we have

$$F \propto \frac{m_1 m_2}{r^2}$$

$$\text{or } F = G\frac{m_1 m_2}{r^2}$$

where G is a constant of proportionality and is called as universal gravitational constant, having always the same value everywhere for all pairs of objects.

This law holds irrespective of the nature of the two objects (size, shape, mass, etc.), at all times and at all places, due to these characteristics this law is known as Newton's universal law of gravitation (G). It is equal to the force of attraction between objects of 1 kg each separated by a distance of 1 m. The accepted numerical value of G is 6.67×10^{-11} Nm^2 kg^{-2}. Its dimensional formula is $[M^{-1}L^3T^{-2}]$, and units in SI system are Nm^2 kg^{-2}.

In vector notation,

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r} = -G\frac{m_1 m_2}{r^3}\vec{r}$$

where $\vec{r} = r\hat{r}$. The negative sign shows that the force is attractive.

Acceleration due to gravity '*g*' : Acceleration produced in a body due to the force of gravity is known as acceleration due to gravity. Its value in SI system is 9.8 m/s^2 and in F P S system is 32 ft/sec^2.

Relation between '*g*', 'G' and Mass of the earth (M_e)

$$\text{Since } F = G\frac{M_e m}{R_e^2} = mg$$

$$\therefore \quad g = \frac{GM_e}{R_e^2}$$

This is the relation between '*g*' and 'G'.

$$\text{Also } M_e = \frac{gR_e^2}{G}$$

where M_e = mass of the earth

R_e = radius of the earth.

The above noted formula have the same shape for all the planets. The value of 'g' is minimum at mercury and maximum at jupiter.

Mean density of the earth : Taking earth to be a sphere of radius R_e and mean density ρ, we get

$$M_e = \frac{4}{3}\pi R_e^3 \rho = \frac{g R_e^2}{G}$$

$$\text{Hence, } \rho = \frac{3g}{4\pi G R_e}$$

Inertial mass and gravitational mass :

Inertial mass : Inertial mass of an object can be defined as the ratio of the magnitude of the external force applied on it to the magnitude of the acceleration produced in it. Hence,

$$m_i = \frac{F}{a}$$

The mass of an object which gives the gravitational pull due to earth's surface, is known as its gravitational mass. Hence,

$$m_g = \frac{F R_e^2}{G M_e}$$

Inertial mass and gravitational mass are taken to be indentical for all practical processes. Newton proved that they are equivalent.

Variation in value of 'g'

Variations in the value of g can be studied under four headings.

(I) Variation in value of 'g' on the surface of earth: Variation in the value of 'g' on the surface of earth is due to two reasons.

***(a)* Due to shape of earth :** The earth is elliptical in shape. It is flatter at the poles and bulged out at the equator. Now, we know that $g \propto \frac{1}{R^2}$, therefore the value of g at the equator is minimum and at the poles is maximum ($\because$ Radius at poles is < Radius at equator).

***(b)* Due to rotation of earth :** Earth rotates about its axis therefore every object placed on the surface of earth also rotates about the same axis. The centripetal force is needed for circular motion. A component of true weight of the body provides this force and the other component is responsible for the observed value of g. If the observed value of g at the lattitude λ is represented by g_λ, then

$$g_\lambda = g_e - R_e \omega^2 \cos^2 \lambda$$

where ω = angular velocity of rotation of the earth.

g_λ = acceleration due to gravity at lattitude λ.

At equator $\lambda = 0°$; $\cos\lambda = 1$

$\therefore g_\lambda = g_e - R_e\omega^2$

At poles $\lambda = 90°$; $\cos\lambda = 0$

$\therefore g_\lambda = g_e$

Thus it is clear that

(i) If the rate of rotation of earth increase, then the value of g will decrease at all places except at the poles.

(ii) Maximum effect of rotation takes place at the equator, if the earth stops rotating then the value of g will increase by a factor $R_e\omega^2$ at equator.

(iii) If the earth starts rotating with an angular speed 17 times the present speed of rotation then the objects will fly off the equator (i.e., g will be zero at the equator).

(iv) If the observed value of g becomes zero at equator, then the length of day will be 1.4 hr.

(II) Variation of '*g*' above the surface of earth:

As we go above the surface of the earth, the value of '*g*' decreases.

(a) The value of '*g*' at a height '*h*' above the surface of the earth is

$$g' = \frac{GM_e}{(R_e + h)^2} = \frac{g_e R_e^2}{(R_e + h)^2}$$

Where 'R_e' is the radius of earth and M_e is the mass of earth, if $h << R_e$, then

$$g' = g_e \left[1 - \frac{2h}{R_e}\right]$$

$\therefore$ the decrease in the above value of g on going up a height 'h' above the surface of earth.

$$\Delta g_e = \frac{2g_e h}{R_e}$$

(III) Variation of 'g' below the surface of earth

(*i*) The value of 'g' decreases as we go below the surface of earth because the effective mass of earth attracting the body decreases.

(*ii*) The value of 'g' at a depth x from the surface of earth is $g' = g\left\{1 - \left(\frac{x}{R}\right)\right\}$. At the centre of earth $x = R$, hence $g = 0$.

(*iii*) The decrease in the value of 'g' at a depth 'x' below the surface of earth is $\Delta g = \left(\frac{gx}{R}\right)$.

(*iv*) The variation in value of 'g' is linear.

Escape velocity

Escape velocity can be defined as the minimum velocity with which an object must be projected from the surface of the earth in order that it may escape the gravitational force of the earth. It is given by

$$V_e = \sqrt{\frac{2GM_e}{R_e}}$$

Clearly, escape velocity is independent of the mass of the object projected. Its value is 11.2 kms^{-1}.

Escape velocities of man at various planets

Planet	Escape velocity ($km\ s^{-1}$)
Moon	2.3
Earth	11.2
Mercury	4.3
Mars	5.0
Venus	10.3
Uranus	22.0
Neptune	24.0
Saturn	36.0
Jupiter	60.0

Satellite velocity (or orbital velocity) : A satellite is one which when projected with a proper velocity, keeps on revolving in a specified orbit. Hence satellite velocity is that velocity with which if an object is projected, it will become the satellite.

If an object of mass m revolves in a fixed circular orbit of radius r around the earth, its orbital velocity is given by,

$$V_s = \sqrt{\frac{GM_e}{r}}$$

If the satellite is revolving at a height h above the surface of the earth, then $r = R_e + h$, hence

$$V_s = \sqrt{\frac{GM_e}{R_e + h}}$$

If $h << R_e$, then

$$V_s = \sqrt{\frac{GM_e}{R_e}} = \frac{V_e}{\sqrt{2}}$$

or escape velocity = $\sqrt{2}$ × satellite velocity

$$\text{Also, } g = \frac{GM_e}{R_e^2}$$

Hence, $V_s = \sqrt{gR_e}$

Its value is about 8 km s^{-1} near the earth's surface.

Time period (or period of revolution) of a satellite : Time period of a satellite is the time taken by the satellite to complete one revolution round the earth. Hence,

$$T = \frac{\text{circumference of the circular orbit}}{\text{orbital velocity}}$$

$$= \frac{2\pi r}{V_s}$$

$$= 2\pi (R_e + h)\sqrt{\frac{R_e + h}{GM_e}}$$

$$= 2\pi\sqrt{\frac{(R_e + h)^3}{GM_e}}$$

Also, $GM_e = gR_e^2$

$$\text{Hence, } T = 2\pi\sqrt{\frac{(R_e + h)^3}{gR_e^2}}$$

If $h << R_e$, i.e., satellite is orbiting very close to the earth, then

$$T = 2\pi\sqrt{\frac{R_e^3}{gR_e^2}} = 2\pi\sqrt{\frac{R_e}{g}}$$

Height of a satellite :

$$\text{Since, } T = 2\pi\sqrt{\frac{(R_e + h)^3}{gR_e^2}}$$

$$\text{Hence, } T^2 = \frac{4\pi^2}{gR_e^2}(R_e + h)^3$$

$$\text{or } (R_e + h)^3 = \frac{gR_e^2 T^2}{4\pi^2}$$

$$\text{or } R_e + h = \left(\frac{gR_e^2 T^2}{4\pi^2}\right)^{1/3}$$

or height of satellite,

$$h = \left(\frac{gR_e^2 T^2}{4\pi^2}\right)^{1/3} - R_e.$$